Contents lists available at Science-Gate

International Journal of Advanced and Applied Sciences

Journal homepage: http://www.science-gate.com/IJAAS.html

The commutativity of prime rings with homoderivations

E. F. Alharfie [1,*], N. M. Muthana [2]

[1]Department of Mathematics, Tabuk University, Tabuk, Saudi Arabia
[2]Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia

ARTICLE INFO

Article history:
Received 4 November 2017
Received in revised form
19 February 2018
Accepted 12 March 2018

Keywords:
Prime ring
Homoderivation
Commutativity theorems

ABSTRACT

Let R be a ring with center $Z(R)$, and I be a nonzero left ideal. An additive mapping $h: R \to R$ is called a homoderivation on R if $h(xy) = h(x)h(y) + h(x)y + xh(y)$for all $x.y \in R$. In this paper, we prove the commutativity of R if any of the following conditions is satisfied for all $x.y \in R$: (i) $xh(y) \pm xy \in Z(R)$. (ii) $xh(y) \pm yx \in Z(R)$. (iii) $xh(y) \pm [x.y] \in Z(R)$ (iv)$[x.y] \in Z(R)$(v)$[h(x)y] \pm xy\ Z(R)$and (vi) $[h(x).y] \pm yx \in Z(R)$. This result is in the sprite of the well-known theorem of the commutativity of prime and semiprime rings with derivations satisfying certain polynomial constraints. Also, we prove that the commutativity of prime ring on R, if R admits a nonzero homoderivation h such that $h([x.y]) = \pm[x.y]$ for all $x.y$ in a nonzero left ideal.

1. Introduction

Throughout, R denotes a ring with a center $Z(R)$. We write $[x.y]\ for\ xy - yx$ and is called the commutator. A ring R is called prime if $aRb = 0$ implies $a = 0\ or\ b = 0$ and is called semiprime if $aRa = 0$ then $a = 0$. A derivation on R is an additive mapping $d: R \to R$ satisfying $d(xy) = d(x)y + xd(y)$ for all $x.y \in R$. El Sofy (2000) defined a homoderivation on R to be an additive mapping h from R into itself such that $h(xy) = h(x)h(y) + h(x)y + xh(y)$ for all $x.y \in R$. The only additive map which is both derivation and homoderivation on prime ring is the zero map. If $S \subseteq R$, then a mapping $f: R \to R$ preserves $S\ if\ f(S) \subseteq S$. A mapping $f: R \to R$ is said to be zero-power valued on S if f preserves S and if for each $x \in S$, there exists a positive integer $n(x) > 1$ such that $f^{n(x)} = 0$ (El Sofy, 2000). Ashraf and Rehman (2001) had shown that if R is a prime ring, I an ideal of R and $d: R \to R$ is a derivation of R, then R is a commutative ring if and only if R satisfies any one of the properties

$d(xy) \pm xy \in Z(R). d(xy) + xy \in Z(R). d(xy) \pm yx \in Z(R). d(xy) + yx \in Z(R). d(x)d(y) \pm xy \in Z(R)\ and\ d(x)d(y) + xy \in Z(R)\ for\ all\ x.y \in I.$

Motivated by these results, we prove a similar result regarding homoderivations. To achieve our aim, we will use the following lemma.

Lemma 1.1 (Lemma 4): Let b and ab be in the center of a prime ring R. If b≠0, then a is in Z(R) (Mayne, 1984).

Remark 1.2 (Remark 3): Let R be a prime ring. If R contains a nonzero commutative left ideal, then R is a commutative ring (Bresar, 1993).

Lemma 1.3 (Lemma 1.1): Let R be a ring and 0≠I a right ideal of R. Suppose that a∈I such that $a^n = 0$ for a fixed integer n. Then R has a nonzero nilpotent ideal (Herstein, 1969).

Lemma 1.4 (Corollary 2.5): Let R be a prime ring of characteristic not 2 and I a nonzero left ideal. If R admits a nonzero homoderivation h which is centralizing on I, then R is commutative.

2. On the commutative conditions

Theorem 2.1: Let R be a prime ring of characteristic not 2, and I be a nonzero left ideal in R. If h is a nonzero homderivation which is zero-power valued on I. Then, for all x, y∈I, the following conditions are equivalent:

i. $xh(y) \pm xy \in Z(R)$
ii. $xh(y) \pm yx \in Z(R)$
iii. $xh(y) \pm [x.y] \in Z(R)$
iv. $h(y)x \pm [x.y] \in Z(R)$
v. $[h(x).y] \pm xy \in Z(R)$
vi. $[h(x).y] \pm yx \in Z(R)$
vii. *R is commutative.*

Proof: If (vii) holds then all other conditions are true. To prove (i) ⟹ (vii). By hypothesis, we have

$$xh(y) \pm xy \in Z(R)\ for\ all\ x, y \in I. \quad (1)$$

* Corresponding Author.
Email Address: ealharfie@ut.edu.sa (E. F. Alharfie)
https://doi.org/10.21833/ijaas.2018.05.010

Replacing x by yx in (1), we get $y(xh(y) \pm xy) \in Z(R)$.

By Lemma 1.1, $y \in Z(R)$ or $xh(y) \pm xy = 0$. If y∈Z(R) for all y∈I hence I⊂Z(R). Therefore I is commutative. By Remark 1.2, R is commutative. If

$$xh(y) \pm xy = 0 \text{ for all } x, y \in I. \quad (2)$$

Replace y by yx in (2), we have x(h(y) ±y)h(x)=0. Since h is zero-power valued on I, there exists an integer n(y)>1 such that $h^{n(y)}(y) = 0$ for all y∈I. Replacing y by

$$y - h(y) + h^2(y) + \cdots + (-1)^{n(y)-1} h^{n(y)-1}(y)$$

in the last relation. We have $xyh(x) = 0$ for all $x, y \in I$. Hence, $xRIh(x) = 0$ for all $x \in I$. But $I \neq 0$. So $Ih(x) = 0$ for all $x \in I$. Hence the Eq. 2 implies $x^2 = 0$ for all $x \in I$. By Lemma 1.3, R has a nonzero nilpotent ideal which contradict that R is prime ring.

To prove (ii) → (vii). By hypothesis, we have

$$xh(y) \pm yx \in Z(R) \text{ for all } x.y \in I. \quad (3)$$

Replace x by yx in (3), we have $y(xh(y) \pm yx) \in Z(R)$. By Lemma 1.1, $y \in Z(R)$ or $xh(y) \pm yx = 0$, If $y \in Z(R)$ for all $y \in I$. hence $I \subseteq Z(R)$. Therefore I is commutative. By Remark 1.2, R is commutative. If

$$xh(y) \pm yx = 0 \text{ for all } x.y \in I. \quad (4)$$

Replace y by xy in (4) we have $xh(x)(h(y) \pm y) = 0$ for all $x.y \in I$. Since h is zero-power valued on I , so $xh(x)y = 0$ for all $x.y \in I$. Hence, $xh(x)RI = 0$ for all $x \in I$ · By primeness of R, we get $xh(x) = 0$ for all $x \in I$. So, by (3) we get $x^2 = 0$ for all $x \in I$. By Lemma 1.3, this is contradiction. To prove (iii) → (vii). By hypothesis, we have

$$xh(y) \pm [x.y] \in Z(R) \text{ for all } x.y \in I. \quad (5)$$

Replace x by yx in (5), $y(xh(y) \pm [x.y]) \in Z(R)$ either $y \in Z(R)$ or $xh(y) \pm [x.y] = 0$. If $y \in Z(R)$ and $I \subseteq Z(R)$ then I is commutative ideal. By Remark 1.2, R is commutative. If

$$xh(y) \pm [x.y] = 0 \text{ for all } x.y \in I. \quad (6)$$

Replace y by yx in (6) we get:

$$x(h(y) \pm y)h(x) = 0 \text{ for all } x.y \in I.$$

Since h is zero-power valued on I. so we get $xyh(x) = 0$ for all $x.y \in I$ which implies $xRIh(x) = 0$ for all $x \in I$. By primeness of R either $x = 0$ or $Ih(x) = 0$. But $I \neq 0$. So $Ih(x) = 0$ for all $x \in I$. Form (6) we have $[x.y] = 0$ for all $x.y \in I$. Then I is commutative ideal. By Remark 1.2. We have R is commutative. To prove (iv) ⟶ (vii) by hypothesis we get:

$$h(y)x \pm [x.y] \in Z(R) \text{ for all } x.y \in I. \quad (7)$$

Replace x by xy in (7)

$$(h(y)x \pm [x.y])y \in Z(R)$$

Since $h(y)x \pm [x.y] \in Z(R)$, we get $(h(y)x \pm [x.y])y = y(h(y)x \pm [x.y])$. So $y(h(y)x \pm [x.y]) \in Z(R)$. By Lemma 1.1, we have $y \in Z(R)$ or $h(y)x \pm [x.y] = 0$. If $y \in Z(R)$ for all $y \in I$. $I \subseteq Z(R)$. Then I is commutative, by Remark 1.2, R is commutative. If

$$h(y)x \pm [x.y] = 0 \text{ for all } x.y \in I. \quad (8)$$

Replace y by xy in (8), $h(x)(h(y) \pm y)x = 0$. Since h is zero-power valued on I, so $h(x)yx = 0$ for all $x.y \in I$. Then we get $h(x)RIx = 0$.

By primeness of R we have $h(x) = 0$ since $Ix \neq 0$ for all $x \in I$.

If $h(x) = 0$ for all $x \in I$, we have $[x.y] = 0$ by (8), then I is commutative. By Remark 1.2, R is commutative.

To prove (v) ⟶ (vii). By hypothesis, we have

$$[h(x).y] \pm xy \in Z(R) \text{ for all } x.y \in I. \quad (9)$$

Replace y by yh(x) in (8) for all $x.y \in I$, we have,

$$([h(x).y] \pm xy)h(x) \in Z(R) \text{ for all } x.y \in I.$$

By Lemma 1.1 either $h(x) \in Z(R)$ or $[h(x).y] + xy = 0$ for all $x.y \in I$.

If $h(x) \in Z(R)$ for all $x \in I$. then $[h(x).x] = 0$. By Lemma 1.4, R is commutative. If

$$[h(x).y] + xy = 0 \text{ for all } x.y \in I. \quad (10)$$

Replace y by xy in (10)

$$[h(x).xy] + xxy = 0 \text{ for all } x.y \in I.$$
$$x([h(x).y] + xy) + [h(x).x]y = 0 \text{ for all } x.y \in I.$$
$$[h(x).x]y = 0 \text{ for all } x.y \in I.$$
$$[h(x).x]RI = 0 \text{ for all } x.y \in I.$$

By primeness of R and $I \neq 0$, we have $[h(x).x] = 0$ for all $x \in I$. By Lemma 1.4, R is commutative.

To prove (vi) ⟶(vii). By hypothesis, we have

$$[h(x).y] \pm yx \in Z(R) \text{ for all } x.y \in I. \quad (11)$$

Replace y by h(x)y in (11)

$$h(x)([h(x).y] + yx) \in Z(R) \text{ for all } x.y \in I.$$

By Lemma 1.1 either $h(x) \in Z(R)$ or $[h(x).y] + yx = 0$.

If $h(x) \in Z(R)$ for all $x \in I$. then $[h(x).x] = 0$. By Lemma 1.4, R is commutative. If

$$[h(x).y] + yx = 0 \text{ for all } x.y \in I \quad (12).$$

Replace y by xy in (12)

$$x([h(x).y] + yx) + [h(x).x]y = 0 \text{ for all } x.y \in I.$$
$$[h(x).x]y = 0 \text{ for all } x.y \in I.$$

Since I is a nonzero left ideal $[h(x).x]RI = 0$. By primeness of R and $I \neq 0$, we have $[h(x).x] = 0$ for all $x \in I$. By Lemma 1.4, R is commutative.

3. On condition $h[x.y] = -[x.y]$

Daif and Bell (1992) proved that a prime ring R with a nonzero ideal I must be commutative if it admits a derivation d such that $d([x.y]) = -[x.y]$. Motivated by their results, we investigate the commutativity of rings admitting a homoderivation h such that $h([x.y]) = -[x.y]$. We begin with the following useful lemma.

Lemma 3.1 (Corollary 3.4.2): Let R be a prime ring of char(R) $\neq 2$. and I $\neq (0)$, a two sides ideal of R. If R admits a a nonzero homoderivation h on I such that $h([x.y]) = [x.y]$ for all $x.y \in I$. Then R is commutative (El Sofy, 2000).

Theorem 3.2: Let I be nonzero left ideal in a prime ring R that admits a homoderivation h which is zero-power valued on I satisfying $xy + h(xy) = yx + h(yx)$ for all $x.y \in I$. Then R is commutative.

Proof: By hypothesis,

$$xy + h(xy) = yx + h(yx) \ \ for\ all\ \ x\,y \in I.$$

i.e.,

$$h([x.y]) = -[x.y]\ \ for\ all\ x.y \in I.$$

Therefore

$$[h(x) + x.h(y)] + [h(x).y] = -[x.y]\ \ for\ all\ \ x.y \in I$$
$$[h(x) + x.h(y)] + [h(x) + x.y] = 0\ \ for\ all\ \ x.y \in I$$
$$[h(x) + x.h(y) + y] = 0\ \ for\ all\ \ x.y \in I.$$

Since h is zero-power valued on I, so there exists an integer $n(y) > 1$ such that $h^{n(y)}(y)=0$ for all $y \in I$. Replacing y by

$$y - h(y) + h^2(y) + \cdots + (-1)^{n(y)-1}h^{n(y)-1}(y)$$

in the last relation. Also, there exists an integer $n(x) > 1$ such that $h^{n(x)}(x)=0$ for all $x \in I$ replacing x by $x-h(x)+h^2(x)+...+(-1)^{n(x)-1}h^{n(x)-1}(x)$ in the last relation, we get

$$[x.y] = 0\ \ for\ all\ \ x.y\ \in I$$

Then I is a commutative ideal in prime ring. By Remark 1.2, R is commutative.

Theorem 3: Let R be a prime ring with char (R) $\neq 2$ and I be a nonzero ideal of R. Suppose h is a nonzero homoderivation which is zero-power valued on I. If one of the following conditions are satisfied for all $x.y\ \in$ I:

i. $h(xy) = xy.$
ii. $h(xy) = yx.$

Then R is commutative.

Proof: Suppose (i) is satisfies for all $x.y \in I$ we get

$$h(xy - yx) = xy - yx\ \ for\ all\ \ x.y \in\ I$$
$$h(xy) - h(yx) = xy - yx\ \ for\ all\ \ x.y \in I$$
$$h([x.y]) = [x.y]\ \ for\ all\ \ x.y \in I$$

By Lemma (3.1), R is commutative. Suppose (ii) is satisfies for all $x.y \in I$. We get

$$h(xy) - h(y\,x) = yx -\ xy\ \ for\ all\ \ x.y \in I$$
$$h([x.y]) = -[x.y]\ \ for\ all\ \ x.y \in I. \quad (14)$$

By Theorem (3.2), we obtain R is commutative.

4. Conclusion

The goal of this paper is to prove the commutativity of prime rings with homoderivation which satisfying some algebraic conditions. This article is divided into two sections; in the first section, the commutativity of prime rings R was proved of the homoderivation on R satisfies following conditions for all

$x.y \in R$: $(i) xh(y) \pm xy \in Z(R)$. $(ii) xh(y) \pm yx \in Z(R)$. $(iii) xh(y) \pm [x.y] \in Z(R)$. $(iv) h(y)x \pm [x.y] \in Z(R)$. $(v) [h(x).y] \pm xy \in\ Z(R)$ and $(vi) [h(x).y] \pm yx \in Z(R)$.

In the second section, we investigate the commutativity of prime ring, if R admits a nonzero homoderivation h such that $h([x.y]) = \pm [x.y]$ for all $x.y$ in a nonzero left ideal.

References

Ashraf M and Rehman NU (2002). On commutativity of rings with derivations. Results in Mathematics, 42(1-2): 3-8.

Bresar M (1993). Centralizing mappings and derivations in prime rings. Journal of Algebra, 156(2): 385-394.

Daif MN and Bell HE (1992). Remarks on derivations on semiprime rings. International Journal of Mathematics and Mathematical Sciences, 15(1): 205-206.

El Sofy MM (2000). Rings with some kinds of mappings. M.Sc. Thesis, Cairo University, Branch of Fayoum, Cairo, Egypt.

Herstein IN (1969). Topics in ring theory. The University of Chicago Press, Chicago, USA.

Mayne JH (1984). Centralizing mappings of prime rings. Canadian Mathematical Bulletin, 27(1): 122-126.